

Unification mechanism for gauge and spacetime symmetries

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Abstract. A group theoretical mechanism for unification of local gauge and spacetime symmetries is introduced. No-go theorems prohibiting such unification are circumvented by slightly relaxing the usual requirement on the gauge group: only the so called Levi factor of the gauge group needs to be compact semisimple, not the entire gauge group. This allows a non-conventional supersymmetry-like extension of the gauge group, glueing together the gauge and spacetime symmetries, but not needing any new exotic gauge particles. It is shown that this new relaxed requirement on the gauge group is nothing but the minimal condition for energy positivity. The mechanism is demonstrated to be mathematically possible and physically plausible on a $U(1)$ based gauge theory setting. The unified group, being an extension of the group of spacetime symmetries, is shown to be different than that of the conventional supersymmetry group, thus overcoming the McGlinn and Coleman-Mandula no-go theorems in a non-supersymmetric way.

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1. Introduction

Unification attempts of internal (gauge) and spacetime symmetries is a long pursued subject in particle field theory. If such unification exists, it would relate coupling factors in the Lagrangian to each-other, which is a strong theoretical motivation. The non-trivialness of the problematics of such unification, however, is well-known. The Coleman-Mandula no-go theorem [1] forbids the most simple unification scenarios. Namely, any larger symmetry group, satisfying a set of plausible properties required by a particle field theory context, and containing the group of spacetime symmetries as a subgroup as well as a gauge group, must be of the trivial form: gauge group \times group of spacetime symmetries[‡]. Also, the earlier theorem of McGlinn [2] concluded in the same direction. The classification result of O’Raifeartaigh [3] on Poincaré group extensions is also usually interpreted in a similar manner. After the discovery of these results, the simple unification attempts of gauge symmetries with spacetime symmetries were not pursued further. Instead, a large amount of research was carried out along the question: can the Poincaré Lie algebra be extended at all in at least by means of some mathematically generalized manner? The answer was positive, as stated by the result of Haag, Lopuszanski and Sohnius [4], and hence the era of supersymmetry (SUSY) was born.

By studying the details of the proof of McGlinn and Coleman-Mandula theorems [5] one finds that the assumption of the presence of a positive definite non-degenerate invariant scalar product on the Lie algebra of the gauge group is essential. Equivalently, these no-go theorems assume that the gauge group is of the form $U(1) \times \dots \times U(1) \times$ a semisimple compact Lie group. The motivations behind this requirement are threefold:

- (i) Group theoretical convenience: the classification of semisimple Lie groups is well understood.
- (ii) Experimental justification: the Standard Model (SM) has a gauge group of the form $U(1) \times SU(2) \times SU(3)$, which satisfies the requirement.
- (iii) Positive energy condition: the energy density expression of a Yang-Mills (gauge) field involves the pertinent invariant scalar product on the Lie algebra of the gauge group, and that is required to be positive definite.

Traditionally, gauge groups not obeying the above rule are believed to violate positive energy condition, and therefore are considered to be unphysical. However, looking more carefully, the positive energy condition merely requires that the invariant scalar product on the Lie algebra of the gauge group must be positive *semidefinite*.

[‡] Whenever a particle field theory model is considered on a fixed flat background spacetime, i.e. not considered as coupled to General Relativity (GR), then the group of spacetime symmetries is simply the Poincaré group. On the other hand, whenever a fully general relativistic field theory is studied, the group of spacetime symmetries is the full diffeomorphism group of the spacetime manifold, acting on the field configurations. Eventually, a general relativistic field theory might be also conformally invariant, in which case the group of spacetime symmetries is the diffeomorphism group along with conformal rescalings (Weyl rescalings) of the spacetime metric tensor field.

In this paper we construct an example when this relaxed condition is considered, and show that this case is mathematically possible, physically plausible, and can be a key to unification of gauge and spacetime symmetries.

2. Structure of Lie groups and supersymmetry

2.1. Levi decomposition theorem

Recall that the symmetry group of flat spacetime, the Poincaré group \mathcal{P} is composed of the group of spacetime translations \mathcal{T} and of the homogeneous Lorentz group \mathcal{L} . Moreover, the group of spacetime translations \mathcal{T} form a *normal subgroup*§ within the Poincaré group \mathcal{P} . Also recall that the Poincaré group can be written as $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$, where \rtimes denotes *semi-direct product*||. It is seen that in the above formula \mathcal{T} is an abelian normal subgroup of \mathcal{P} , and that the subgroup \mathcal{L} of \mathcal{P} is a simple matrix group. The Levi decomposition theorem [6] states that such decomposition property is generic to all Lie groups. Namely, any Lie group, assumed now to be connected and simply connected for simplicity, has the structure $R \rtimes L$, R being a solvable normal subgroup called the *radical* and L being a semisimple subgroup called the *Levi factor*. The *semisimpleness* of L means that the *Killing form* $(x, y) \mapsto \text{Tr}(\text{ad}_x \text{ad}_y)$ is non-degenerate on the Lie algebra of L , using the symbol $\text{ad}_x(\cdot) := [x, \cdot]$ for any Lie algebra element x . The *solvability* of R means that it represents the degenerate directions of the Killing form. It may also be formulated in terms of an equivalent property: for the Lie algebra r of R with the definition $r^0 := r$, $r^1 := [r^0, r^0]$, $r^2 := [r^1, r^1]$, \dots , $r^k := [r^{k-1}, r^{k-1}]$, \dots , one has $r^k = \{0\}$ for finite k . A special case is when the radical R is said to be *nilpotent*: there exists a finite k for which for all $x_1, \dots, x_k \in r$ one has $\text{ad}_{x_1} \dots \text{ad}_{x_k} = 0$. An even more special case is when the radical R is *abelian*: for all $x \in r$, one has $\text{ad}_x = 0$.

The (proper) Poincaré group with its structure $\mathcal{T} \rtimes \mathcal{L}$ is a demonstration of Levi decomposition theorem, where \mathcal{T} is the abelian normal subgroup consisting of spacetime translations, being the radical, and where \mathcal{L} is the semisimple subgroup consisting of the (proper) homogeneous Lorentz transformations, being the Levi factor. Groups like $\text{SU}(N)$, often turning up as gauge groups in Yang-Mills models, however are semisimple, and therefore their radical vanishes, i.e. such a group consists purely of its Levi factor. Historically, groups with nonvanishing radical are usually not studied in context with physical field theory models, even though the symmetry group of flat spacetime readily provides an archetypical example for such groups.

§ A subgroup N within a larger group is called normal subgroup whenever it is invariant to the adjoint action of the larger group, i.e. whenever one has $g N g^{-1} \subset N$ for all elements g of the larger group.

|| Semi-direct product means that any element of the larger group can uniquely be written as a product of elements from the coefficient groups, and that at least the leftmost coefficient group is normal subgroup. The two coefficient groups are not required to commute. When they commute, then also the rightmost coefficient group is normal subgroup, and the semi-direct product becomes a direct product, denoted by \times .

2.2. Levi structure of supersymmetry group

The Levi decomposition theorem also sheds a light on the group structure of supersymmetry transformations, being an extension of the Poincaré group. That Lie group has a Levi decomposition of the form $\mathcal{S} \rtimes \mathcal{L}$, where \mathcal{S} is the nilpotent normal subgroup consisting of *supertranslations*, being the radical, and where \mathcal{L} is the semisimple subgroup consisting of the (proper) homogeneous Lorentz transformations, being the Levi factor. The supertranslations are defined as transformations on the vector bundle of superfields [7, 8, 9]. With supertranslation parameters (ϵ^A, d^a) they are of the form

$$\begin{pmatrix} \theta^A \\ x^a \end{pmatrix} \mapsto \begin{pmatrix} \theta^A + \epsilon^A \\ x^a + d^a + \sigma_{AA'}^a i(\theta^A \bar{\epsilon}^{A'} - \epsilon^A \bar{\theta}^{A'}) \end{pmatrix} \quad (1)$$

in terms of “supercoordinates” (Grassmann valued two-spinors) and affine spacetime coordinates.[¶] From Eq.(1) it is seen that although the pure spacetime translations \mathcal{T} form an abelian normal subgroup inside \mathcal{S} , but $\mathcal{S} \neq \mathcal{T} \rtimes \{\text{some other subgroup}\}$, and thus such splitting is not applicable for the entire supersymmetry group. A geometric consequence of that phenomenon is illustrated in Figure 1: a pure supertranslation with parameter $(\epsilon^A, 0)$ does not act *pointwise* (or *fibrewise*), but it transforms a superfield value at a point of spacetime to an other superfield value over a point shifted by a corresponding spacetime translation.

In this paper, however, we shall present a different nontrivial Poincaré group extension, enlarged both on the side of the radical and of the Levi factor, containing both the gauge and the spacetime symmetries, and being of the form

$$\mathcal{T} \rtimes \{\text{some group acting at points of spacetime}\}, \quad (2)$$

and thus rather acting pointwise, similarly as conventional gauge groups do, as illustrated in the left panel of Figure 1.

2.3. O’Raifeartaigh classification of Poincaré group extensions

Let us take a larger symmetry group E with its Levi decomposition $E = R \rtimes L$, containing the Poincaré group $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$ as a subgroup. Then the theorem of O’Raifeartaigh [3] states that either one has $\mathcal{T} \subset R$ and $\mathcal{L} \subset L$ (radical embedded into radical, Levi factor embedded into Levi factor), or one has $\mathcal{T} \rtimes \mathcal{L} \subset L$ (the entire Poincaré group is embedded into the Levi factor of a much larger symmetry group). This result leads to the following classification theorem of O’Raifeartaigh [3] on the possible extensions of the Poincaré group:

[¶] A note about the presentation of supersymmetry transformations: usually, they are presented in the infinitesimal form and in a parametrization which is often referred to as a “graded Lie algebra”, or “super Lie algebra”. That form, however, may be reparametrized in order to form a conventional Lie algebra, as shown in [7, 8, 9]. This Lie algebra presentation, when exponentiated, shall form a conventional Lie group discussed above. This simple reparametrization, although is known in the literature [7, 8, 9], is mostly not used in the traditional way of presentation of SUSY.

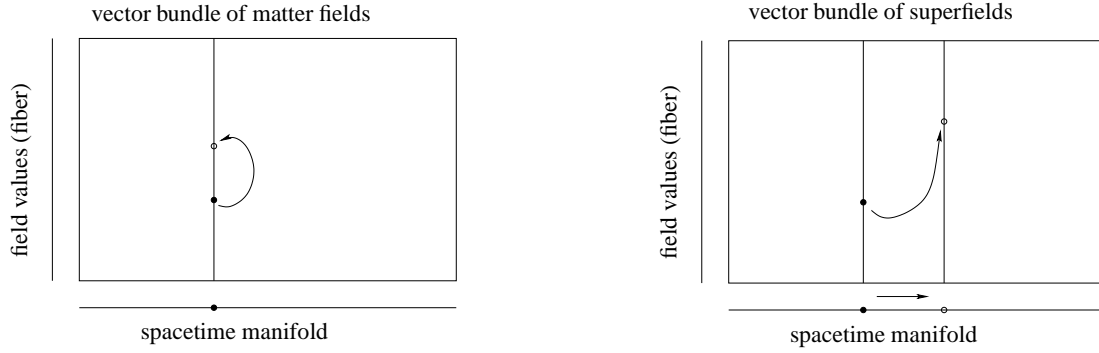


Figure 1. Left panel: Illustration of how in a conventional gauge theory the non-spacetime symmetries (the gauge symmetries) act on the vector bundle of matter fields. The action of the such transformations do preserve the spacetime points, i.e. they act “pointwise” on the matter fields. Right panel: Illustration of how in a supersymmetric theory the non-spacetime symmetries (the pure supertranslations) act on the vector bundle of superfields. The action of such transformations is not “pointwise”, but maps a field value into a field value over a shifted point of spacetime.

- (i) $R = \mathcal{T}$, and $L = \{\text{some semisimple Lie group}\} \times \mathcal{L}$. This means that whenever the radical R of the larger symmetry group solely consists of the spacetime translations, then one has only the trivial group extension $E = \mathcal{P} \times \{\text{some extra symmetries}\}$. This group theoretical phenomenon drives the no-go theorems of McGlinn and Coleman-Mandula.
- (ii) R is an abelian extension of \mathcal{T} , and $\mathcal{L} \subset L$. This means that in the radical R of the larger symmetry group one has the spacetime translations and some abelian extension. The Levi factor L of the extended symmetries E may be larger than \mathcal{L} .
- (iii) R is a non-abelian extension of \mathcal{T} , and $\mathcal{L} \subset L$. In this case the radical R contains the spacetime translations and some non-abelian solvable extension. The Levi factor L of the extended symmetries E can be larger than \mathcal{L} . SUSY and the example to be presented in this paper falls into this case.
- (iv) $\mathcal{T} \rtimes \mathcal{L} \subset L$ and L is a simple Lie group. This case would mean that the Poincaré group is fully embedded into a much larger simple Lie group. Although such a scenario is not shown to be mathematically impossible, but physically is a rather artificial case, and no popular examples are known for such a setting.

Consequently: for non-trivially extending the Poincaré group, its radical must necessarily be extended, as shown by cases (ii)–(iii).

It is seen that the supersymmetry group is of type (iii) in the classification theorem of O’Raifeartaigh: its radical is extended and therefore the no-go theorems of McGlinn and Coleman-Mandula are not applicable. The unification mechanism for gauge and spacetime symmetries proposed in the followings uses the same group theoretical possibility as well, but in a very different way in comparison to SUSY: our extended group shall have the structure Eq.(2), which is not the case for the SUSY group.

3. Unification mechanism for gauge and spacetime symmetries

In terms of global symmetries, our proposed unification mechanism for gauge and spacetime symmetries assumes a structure

$$\begin{array}{c}
 \begin{array}{c} \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \end{array} \\
 \left(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\mathcal{N}}_{\text{solvable internal}} \right) \rtimes \left(\underbrace{\mathcal{G}}_{\text{compact internal}} \times \underbrace{\mathcal{L}}_{\text{(conformal) Lorentz group}} \right) \\
 \underbrace{\hspace{15em}}_{\text{full gauge group}} \\
 \underbrace{\hspace{15em}}_{\text{global symmetries of matter fields when considered over flat spacetime}}
 \end{array} \tag{3}$$

for the unified group. Here, \mathcal{G} symbolizes the usual compact gauge group, being $U(1) \times SU(2) \times SU(3)$ in case of SM, \mathcal{L} denotes the homogeneous part of the spacetime symmetry group, being the homogeneous (possibly conformal) Lorentz group, and \mathcal{N} stands for a non-usual extension of the group of internal symmetries, allowed to be a solvable normal subgroup. The arrows indicate which subgroup acts nontrivially on which normal subgroup, i.e. subgroups not connected by arrows do commute, whereas the others do not. Clearly, such group structure as a Poincaré group extension is potentially allowed by the case (iii) of O’Raifeartaigh classification theorem. Using the semi-associativity of \rtimes and \times , the global unified group described by Eq.(3) can be rewritten in an equivalent form

$$\begin{array}{c}
 \begin{array}{c} \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \end{array} \\
 \underbrace{\mathcal{T}}_{\text{translations}} \rtimes \left(\underbrace{\mathcal{N}}_{\text{solvable internal}} \rtimes \left(\underbrace{\mathcal{G}}_{\text{compact internal}} \times \underbrace{\mathcal{L}}_{\text{(conformal) Lorentz group}} \right) \right) \\
 \underbrace{\hspace{15em}}_{\text{full gauge group}} \\
 \underbrace{\hspace{15em}}_{\text{symmetries of matter fields at points of spacetime}} \\
 \underbrace{\hspace{15em}}_{\text{global symmetries of matter fields when considered over flat spacetime}}
 \end{array} \tag{4}$$

which shows that our unified group, as global symmetries, are of the form of Eq.(2). That naturally motivates to search for a local unified group of gauge and spacetime symmetries in the form

$$\begin{array}{c}
 \begin{array}{c} \downarrow \qquad \downarrow \qquad \downarrow \end{array} \\
 \underbrace{\mathcal{N}}_{\text{solvable internal}} \rtimes \left(\underbrace{\mathcal{G}}_{\text{compact internal}} \times \underbrace{\mathcal{L}}_{\text{(conformal) Lorentz group}} \right) \\
 \underbrace{\hspace{15em}}_{\text{full gauge group}} \\
 \underbrace{\hspace{15em}}_{\text{local symmetries of matter fields at points of spacetime}}
 \end{array} \tag{5}$$

which is just Eq.(4) without the translations, acting as local symmetry on the matter

fields independently at each point of a spacetime manifold. Again, using the semi-associativity of \rtimes and \times , the local unified group described by Eq.(5) can be rewritten in the equivalent form

$$\begin{array}{c}
 \downarrow \\
 \left(\underbrace{\mathcal{N} \rtimes \mathcal{G}}_{\text{full gauge group}} \right) \rtimes \underbrace{\mathcal{L}}_{\text{(conformal) Lorentz group}} \\
 \underbrace{\hspace{15em}}_{\text{local symmetries of matter fields at points of spacetime}}
 \end{array} \tag{6}$$

which implies that there exists a *homomorphism*⁺ from the local unified group Eq.(6) onto the local group of spacetime symmetries \mathcal{L} , and the kernel of that homomorphism is the local group of internal (gauge) symmetries $\mathcal{N} \rtimes \mathcal{G}$. This finding implies the following consequences:

- (i) The full local unified group Eq.(5) has a four-vector representation through the homomorphism onto \mathcal{L} .
- (ii) The group of local internal (gauge) symmetries $\mathcal{N} \rtimes \mathcal{G}$ act trivially on such four-vector representation — hence the name: they act trivially on the spacetime vectors.
- (iii) The full local unified group Eq.(5) acts as the (conformal) Lorentz group on such four-vector representation.
- (iv) Because of the previous point, there exists a uniquely determined Lorentz metric conformal equivalence class on the four-vector representation, preserved by the local unified group Eq.(5).
- (v) Because of the previous point, there exists a uniquely determined Lorentz causal structure preserved by the local unified group Eq.(5).
- (vi) Due to the presence of \mathcal{N} , the local unified group Eq.(5) is indecomposable, i.e. is not of the form of a direct product.

In conclusion, Eq.(5) shows that the local gauge group and the group of local spacetime symmetries would decompose into a direct product $\mathcal{G} \times \mathcal{L}$ as dictated by the McGlinn and Coleman-Mandula no-go theorems, however the solvable normal subgroup \mathcal{N} of local gauge symmetries glues them together, making the unification. With that, the full local gauge group shall be an extended one, $\mathcal{N} \rtimes \mathcal{G}$, as a price to pay. Since \mathcal{N} represents the degenerate directions of the Killing form over the full gauge group $\mathcal{N} \rtimes \mathcal{G}$, it only adds some zero-energy gauge field modes to a field theoretical model having local unified symmetries as Eq.(5). These zero-energy gauge field modes shall also have vanishing Yang-Mills kinetic Lagrangian term, and therefore such unification mechanism does not cost adding new propagating gauge particle fields to the system. They do contribute, however, to other parts of the Lagrangian involving matter fields and

⁺ Group homomorphism: a product preserving mapping from one group to another.

their covariant derivatives, restricting the forms of possible Lagrangians compatible with the extended symmetry requirement. It is remarkable, that the proposed unification mechanism does not necessarily need a breaking of the large symmetry group, as the non-conventional part \mathcal{N} of internal symmetries is inapparent in terms of detectable gauge particles. Also, one should note that the allowed more relaxed structure $\mathcal{N} \rtimes \mathcal{G}$ of the full gauge group means a softer regularity condition than traditionally required in gauge theory: only the Levi factor of the gauge group needs to be compact semisimple, not the entire gauge group itself. This is equivalent to the positive semidefiniteness of the Killing form on the gauge group, and hence is the minimal requirement for the non-negativity of the energy density expression of the Yang-Mills fields in a system with such unified symmetries.

In the coming section we shall construct a minimal version of a unified local symmetry group as in Eq.(5), with $\mathcal{G} = \text{U}(1)$. There is strong indication that the same mechanism can also be performed for the full SM gauge group using the approach of [10].

4. Concrete example for the $\text{U}(1)$ case

Our example for a local unified symmetry group having the structure like Eq.(5) with $\mathcal{G} = \text{U}(1)$ shall be described below. It is a non-supersymmetric extension of the (proper) homogeneous conformal Lorentz group. It is detailed in [11, 12] and in Appendix A.

Let A be a finite dimensional complex unital associative algebra, with its unit denoted by 1 . Whenever A is also equipped with a conjugate-linear involution $(\cdot)^+ : A \rightarrow A$ such that for all $x, y \in A$ one has $(xy)^+ = x^+y^+$, then it shall be called a $^+$ -algebra. Note that this notion differs from the well-known mathematical notion of * -algebra as here the $^+$ -adjoining does not exchange the order of products. Let now A be a finite dimensional complex associative algebra with unit, being also $^+$ -algebra, and possessing a minimal generator system (e_1, e_2, e_3, e_4) obeying the identity

$$\begin{aligned} e_i e_j + e_j e_i &= 0 \quad (i, j \in \{1, 2\} \text{ or } i, j \in \{3, 4\}), \\ e_i e_j - e_j e_i &= 0 \quad (i \in \{1, 2\} \text{ and } j \in \{3, 4\}), \\ e_3 &= e_1^+, \\ e_4 &= e_2^+, \\ e_{i_1} e_{i_2} \dots e_{i_k} &\quad (1 \leq i_1 < i_2 < \dots < i_k \leq 4, \ 0 \leq k \leq 4) \\ &\text{are linearly independent.} \end{aligned} \tag{7}$$

Then we call A *spin algebra*, and we call a minimal generator system obeying Eq.(7) a *canonical generator system*, whereas the $^+$ -operation is called *charge conjugation*. That is, spin algebra is a freely generated unital complex associative algebra with four generators, and the generators admit two sectors within which the generators anticommute, whereas the two sectors commute with each-other, and are charge conjugate to each-other. It is easy to check that if S^* is a complex two dimensional vector space (called the *cospinor space*), and \bar{S}^* is its complex conjugate vector space, then

$\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ naturally becomes spin algebra, where $\Lambda(\cdot)$ denotes the exterior algebra of its argument. It is also seen that any spin algebra is isomorphic (not naturally) to this algebra, i.e. they all have the same structure, but there is a freedom in matching the canonical generators. Some properties of the pertinent mathematical structure is listed in [11]. In terms of a formal quantum field theory (QFT) analogy, the spin algebra can be regarded as the creation operator algebra of a fermion particle with two internal degrees of freedom along with its antiparticle, at a fixed point of spacetime, or equivalently, at a fixed point of momentum space. It is important to understand, however, that in this construction the creation operators of antiparticles are not yet identified with the annihilation operators of particles, i.e. it is not a canonical anticommutation relation (CAR) algebra. As such, the spin algebra reflects the following physical picture:

- (i) The basic ingredients of the system are particles obeying Pauli's exclusion principle.
- (ii) These particles have finite (two) internal degrees of freedom.
- (iii) Corresponding charge conjugate particles are present in the system.

Our extension of the homogeneous conformal Lorentz group shall be nothing but $\text{Aut}(A)$, the *automorphism group* of the spin algebra A . That consists of those invertible $A \rightarrow A$ linear transformations, which preserve the algebraic product as well as the charge conjugation operation.

It can be shown that if the discrete symmetries are omitted, i.e. if the unit connected component of $\text{Aut}(A)$ is considered, then it has a structure of the form

$$\underbrace{\underbrace{\underbrace{N}_{\text{nilpotent internal}} \rtimes \left(\underbrace{U(1)}_{\text{compact internal}} \times \underbrace{\mathcal{L}}_{\text{conformal Lorentz group}} \right)}_{\text{full gauge (internal) symmetries}}}_{\text{symmetries of } A\text{-valued fields at a point of spacetime or momentum space}} \quad (8)$$

which exactly has a structure like Eq.(5). For details we refer to Appendix A and [11, 12]. The nilpotent normal subgroup N of internal symmetries transform a system of canonical generators in such a way, that it adds higher polynomials of the generators to pure generators, and hence they are named “dressing transformations”.

5. Concluding remarks

A unification mechanism for local gauge and spacetime symmetries was presented. The key ingredient is to allow a solvable normal subgroup in the full gauge group, and to only require the Levi factor of the full gauge group to be compact semisimple, not the entire gauge group. This relaxed regularity property of allowed gauge groups is the minimal requirement for energy positivity. The solvable extension of the gauge group is seen not to introduce new propagating gauge boson degrees of freedom, which would contradict present experimental understanding. It is rather seen to be a set of inapparent symmetries, representing “dressing transformations” for pure one-particle

states in a formal quantum field theory setting. The unification mechanism also provides an example for a non-supersymmetric extension of the group of spacetime symmetries, circumventing the McGlinn and Coleman-Mandula no-go theorems in a non-SUSY way. Therefore, the construction of invariant Lagrangians to such a local unified symmetry group is worth to study. That involves representation theory of non-semisimple Lie groups, which is a contemporary branch of research in group theory.

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Appendix A. Details of the concrete example for the U(1) case

The spin algebra A has several important linear subspaces. Given a canonical generator system (e_1, e_2, e_1^+, e_2^+) of A , the followings can be defined: $\Lambda_{\bar{p}q}$ are the linear subspaces of p, q -forms, i.e. the polynomials consisting of p powers of $\{e_1, e_2\}$ and q powers of $\{e_1^+, e_2^+\}$ ($p, q \in \{0, 1, 2\}$), and one has $A = \bigoplus_{p,q=0}^2 \Lambda_{\bar{p}q}$, called to be the $\mathbb{Z} \times \mathbb{Z}$ -grading of A . Then, there are the linear subspaces of k -forms, Λ_k , i.e. the polynomials consisting of k powers of $\{e_1, e_2, e_1^+, e_2^+\}$ ($k \in \{0, 1, 2, 3, 4\}$), and one has $A = \bigoplus_{k=0}^4 \Lambda_k$, called to be the \mathbb{Z} -grading of A . Finally, there are the subspaces Λ_{ev} and Λ_{od} being the even and odd polynomials of $\{e_1, e_2, e_1^+, e_2^+\}$, and one has $A = \Lambda_{\text{ev}} \oplus \Lambda_{\text{od}}$, called to be the \mathbb{Z}_2 -grading of A . The subspace $B := \Lambda_{\bar{0}0} = \mathbb{C}1$ of zero-forms and the subspace $M := \bigoplus_{k=1}^4 \Lambda_k$ of at-least-1-forms shall play an important role as well, and one has $A = B \oplus M$. B is a one-dimensional unital associative subalgebra of A , spanned by the unity and called the *unit algebra*, whereas M is the so called *maximal ideal* of A . An other important subspace is $Z = \Lambda_{\bar{0}0} \oplus \Lambda_{\bar{2}0} \oplus \Lambda_{\bar{0}2} \oplus \Lambda_{\bar{2}2}$, the *center* of A , being the largest unital associative subalgebra in A commuting with all elements of A . All these are illustrated in Figure A1.

In order to study the structure of $\text{Aut}(A)$, it is important to note that an element of $\text{Aut}(A)$ maps a canonical generator system to a canonical generator system, and that an element of $\text{Aut}(A)$ can be uniquely characterized by its group action on an arbitrary preferred canonical generator system. Let us take such a system (e_1, e_2, e_1^+, e_2^+) , with occasional notation $e_3 = e_1^+$, $e_4 = e_2^+$. The group structure of $\text{Aut}(A)$ can then be characterized with the following four subgroups:

- (i) Let $\text{Aut}_{\mathbb{Z} \times \mathbb{Z}}(A)$ be the group of $\mathbb{Z} \times \mathbb{Z}$ -grading preserving automorphisms: they act on the canonical generators as $e_i \mapsto \sum_{j=1}^2 \alpha_{ij} e_j$ and $e_i^+ \mapsto \sum_{j=1}^2 \bar{\alpha}_{ij} e_j^+$ ($i \in \{1, 2\}$),

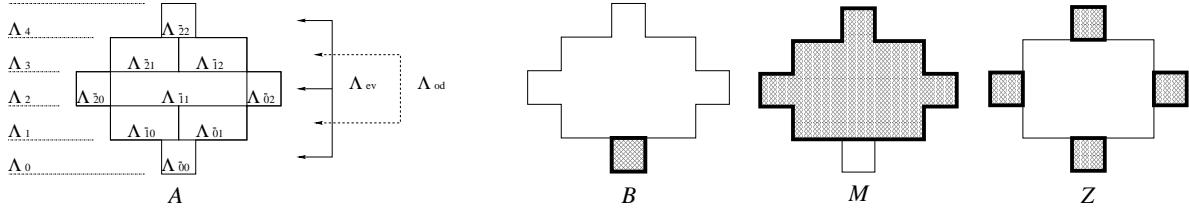


Figure A1. Leftmost panel: illustration of the $\mathbb{Z} \times \mathbb{Z}$, \mathbb{Z} and \mathbb{Z}_2 grading structure of the spin algebra A . The unit element $\mathbb{1}$ resides in the subspace Λ_{00} , whereas the canonical generators span the subspace $\Lambda_{10} \oplus \Lambda_{01}$. Other panels: illustration of the important subspaces of the spin algebra, namely the unit subalgebra B , the maximal ideal M , and the center Z . One unit box depicts one complex dimension on all panels, shaded regions depict the subspaces B , M and Z , respectively.

the bar $(\bar{\cdot})$ meaning complex conjugation and the 2×2 complex matrix $(\alpha_{ij})_{i,j \in \{1,2\}}$ being invertible.

- (ii) Let $\mathcal{J} := \{I, J\}$ be the two element subgroup of \mathbb{Z} -grading preserving automorphisms, I being the identity and J being the involutive complex-linear operator of *particle-antiparticle label exchanging* acting as $e_1 \mapsto e_3$, $e_2 \mapsto e_4$, $e_3 \mapsto e_1$, $e_4 \mapsto e_2$.
- (iii) Let \tilde{N}_{ev} be a subgroup of the \mathbb{Z}_2 -grading preserving automorphisms defined by the relations $e_i \mapsto e_i + b_i$ and $e_i^+ \mapsto e_i^+ + b_i^+$ with uniquely determined parameters $b_i \in \Lambda_{\bar{1}2}$ ($i \in \{1, 2\}$).
- (iv) Let $\text{InAut}(A)$ be the subgroup of inner automorphisms, i.e. the ones of the form $\exp(a)(\cdot)\exp(a)^{-1}$ with some $a \in \text{Re}(A)$. These are of the form $e_i \mapsto e_i + [a, e_i] + \frac{1}{2}[a, [a, e_i]]$ ($i \in \{1, 2, 3, 4\}$) with uniquely determined parameter $a \in \text{Re}(\Lambda_{\bar{1}0} \oplus \Lambda_{\bar{0}1} \oplus \Lambda_{\bar{1}1} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2})$.

With these, the semi-direct product splitting

$$\text{Aut}(A) = \underbrace{\text{InAut}(A) \rtimes \tilde{N}_{ev}}_{=: N} \rtimes \underbrace{\text{Aut}_{\mathbb{Z} \times \mathbb{Z}}(A)}_{=: \text{Aut}_{\mathbb{Z}}(A)} \rtimes \mathcal{J} \quad (\text{A.1})$$

holds. It is seen that a \mathbb{Z} -grading almost determines the underlying $\mathbb{Z} \times \mathbb{Z}$ -grading: only the two-element discrete group of label exchanging transformations \mathcal{J} introduces an ambiguity. The subgroup N shall be called the group of *dressing transformations*, being a nilpotent normal subgroup of $\text{Aut}(A)$. These transformations are mixing higher forms to lower forms, i.e. do not preserve the \mathbb{Z} and \mathbb{Z}_2 -grading defined by our preferred canonical generator system: they map a system of canonical generators like $e_i \mapsto e_i + \beta_i$, the elements β_i residing in the space of at-least-2-forms M^2 ($i \in \{1, 2, 3, 4\}$), deforming the original \mathbb{Z} and \mathbb{Z}_2 -grading to an other one. By direct substitution it is seen that the transformations (i)–(iv) indeed define independent subgroups of $\text{Aut}(A)$, however the proof of decomposition theorem Eq.(A.1) needs a bit more complex mathematical apparatus [12]. The principle of the proof is motivated by [13], studying

the automorphism group of ordinary finite dimensional complex Grassmann (exterior) algebras.

By scrutinizing the subgroups, it is seen that the group \mathcal{J} of label exchanging transformations has the structure of \mathbb{Z}_2 . On the other hand, one has

$$\text{Aut}_{\mathbb{Z} \times \mathbb{Z}}(A) \equiv \text{GL}(2, \mathbb{C}) \equiv \text{U}(1) \times \text{D}(1) \times \text{SL}(2, \mathbb{C}), \quad (\text{A.2})$$

where $\text{D}(1)$ is the dilatation group, i.e. \mathbb{R}^+ with the real multiplication. Note that $\text{D}(1) \times \text{SL}(2, \mathbb{C})$ is nothing but the universal covering group of the (proper) homogeneous conformal Lorentz group. As far as a fixed $\mathbb{Z} \times \mathbb{Z}$ -grading is taken, A can be always represented via ordinary two-spinor calculus, and the algebra identification $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ can greatly ease the calculations due to well-known identities in that formalism [14, 15]. The group of dressing transformations N , however, does not fit automatically into that framework: it needs the proper apparatus of the introduced spin algebra formalism, or care is needed when represented in terms of two-spinors.

Appendix A.1. Important representations of the example group

Due to the presence of the nilpotent normal subgroup N , $\text{Aut}(A)$ is not semisimple. As a consequence, there can be nontrivial invariant subspaces even in the defining representation, i.e. when $\text{Aut}(A)$ acts on A . However, for the same reason, the existence of an invariant subspace in a representation of $\text{Aut}(A)$ does not imply the existence of an invariant complement. The indecomposable $\text{Aut}(A)$ -invariant subspaces of A are listed and illustrated in Figure A2. The invariance of these is seen via the orbits of the subspaces $\Lambda_{\bar{p}q}$ ($p, q \in \{0, 1, 2\}$) by the group action of \mathcal{J} and of N .

The group $\text{Aut}(A)$ naturally acts on A^* , the dual vector space of the spin algebra A with the transpose group action. It may be easily seen that the $\text{Aut}(A)$ -invariant subspaces of A^* can be obtained as annihilators of $\text{Aut}(A)$ -invariant subspaces of A itself.* The indecomposable $\text{Aut}(A)$ -invariant subspaces of A^* are listed and illustrated in Figure A3.

In Figure A3 it is seen that the $\text{Aut}(A)$ -invariant subspace

$$\text{Ann}(B \oplus V) \equiv \Lambda_{11}^* \quad (\text{A.3})$$

is nothing but a four-vector representation of $\text{Aut}(A)$, on which $\text{Aut}(A)$ acts as the homogeneous conformal Lorentz group. In the two-spinor representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ one has simply $\Lambda_{11}^* \equiv \bar{S} \otimes S$. The kernel of the corresponding homomorphism of $\text{Aut}(A)$ onto the homogeneous conformal Lorentz group is said to be the *full gauge group*, having the structure $N \rtimes \text{U}(1)$. Given a four dimensional real vector space T , any injection $T \rightarrow \text{Re}(\Lambda_{11}^*)$ is called a *Pauli injection*, which is the analogue of the “soldering form” in the traditional two-spinor calculus [14, 15], extending the group action of $\text{Aut}(A)$ onto the real four dimensional vector space T . In the usual Penrose abstract index notation that is nothing but the usual mapping $\sigma_a^{AA'}$ between spacetime

* Given a linear subspace $X \subset A$, its annihilator subspace $\text{Ann}(X) \subset A^*$ is the set of all A^* elements which maps the subspace X to zero.

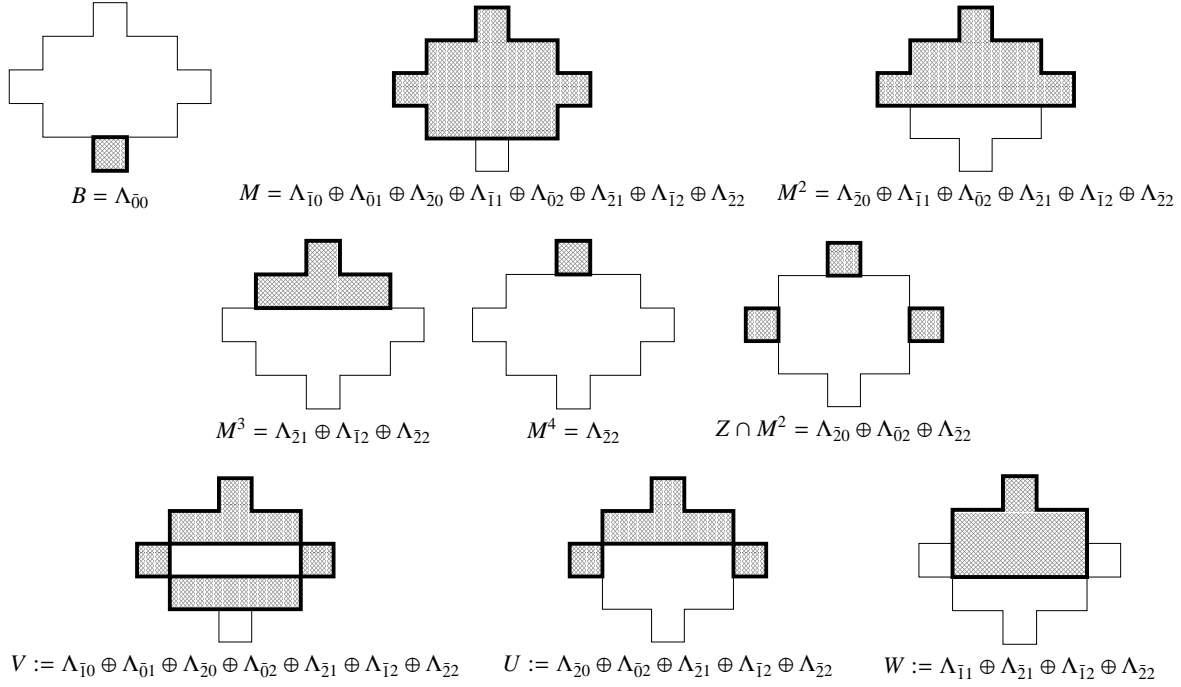


Figure A2. Illustration of the $\text{Aut}(A)$ -invariant indecomposable subspaces of the spin algebra A . One unit box depicts one complex dimension, shaded regions denote the invariant subspaces on all panels.

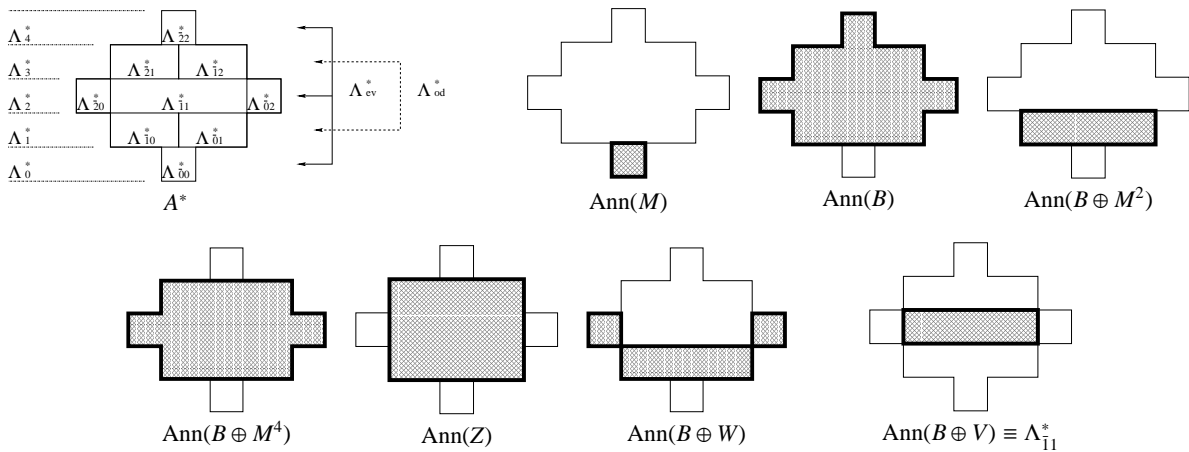


Figure A3. Top left panel: illustration of the $\mathbb{Z} \times \mathbb{Z}$, \mathbb{Z} and \mathbb{Z}_2 grading structure of the dual vector space A^* of the spin algebra A . Other panels: illustration of the $\text{Aut}(A)$ -invariant indecomposable subspaces of the dual vector space A^* of the spin algebra A . One unit box depicts one complex dimension, shaded regions denote the invariant subspaces on all panels. Note that the subspace $\text{Ann}(B \oplus V) \equiv \Lambda_{\bar{1}1}^*$, illustrated on the bottom right panel, is a four-vector representation of $\text{Aut}(A)$ and the pertinent group acts there as the homogeneous conformal Lorentz group.

vectors T and hermitian mixed spinor-tensors $\text{Re}(\bar{S} \otimes S)$. It is seen that the group of dressing transformations N respects this basic relation of two-spinor calculus and hence realizes the group action of $\text{Aut}(A)$ on the spacetime vectors T as the homogeneous conformal Lorentz group.

From Eq.(A.1) it is seen that the connected component $\text{Aut}_0(A)$ of our concrete example $\text{Aut}(A)$ has the group structure

$$\underbrace{\underbrace{\underbrace{N}_{\text{dressing transformations}} \times \left(\underbrace{U(1)}_{\text{internal}} \times \underbrace{D(1) \times \text{SL}(2, \mathbb{C})}_{\text{spacetime related}} \right)}_{\text{full gauge group}}}_{\text{symmetries of } A\text{-valued fields at a point of spacetime or momentum space}} \quad (\text{A.4})$$

which indeed follows the pattern of Eq.(5), providing a demonstrative example of the proposed unification mechanism.

Appendix A.2. Adding the translation or diffeomorphism group

Adding translations to the presented homogeneous conformal Lorentz group extension is trivial. One simply takes a four dimensional real affine space \mathcal{M} as the model of the flat spacetime manifold, with underlying vector space (“tangent space”) T . One takes in addition the spin algebra A , and constructs the trivial vector bundle $\mathcal{M} \times A$. The algebraic product on A extends to the sections of this vector bundle (i.e. to the A -valued fields) pointwise, being translationally invariant. Given a Pauli injection (soldering form) between T and $\text{Re}(\Lambda_{11}^*)$, $\text{Aut}(A)$ acts on T as the homogeneous conformal Lorentz group. The vector bundle automorphisms of $\mathcal{M} \times A$ preserving the algebraic product of fields as well as preserving the Pauli injection shall have the desired group structure including both the spacetime translations and $\text{Aut}(A)$ in a semi-direct product:

$$\underbrace{\left(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\underbrace{N}_{\text{dressing transformations}}}_{\text{full gauge group}} \right) \times \left(\underbrace{U(1)}_{\text{internal}} \times \underbrace{D(1) \times \text{SL}(2, \mathbb{C})}_{\text{spacetime related}} \right)}_{\text{global symmetries of } A\text{-valued fields when considered over flat spacetime}} \quad (\text{A.5})$$

as a global symmetry of fields, following the pattern of Eq.(3). When acting on \mathcal{M} , it shall act as the Poincaré group combined with global metric rescalings. This also implies a causal structure on \mathcal{M} . Clearly, Eq.(A.5) is a non-supersymmetric extension of the Poincaré group, circumventing McGlinn and Coleman-Mandula no-go theorems.

The “gauging” of $\text{Aut}(A)$, i.e. making $\text{Aut}(A)$ a local symmetry is also trivial. Let \mathcal{M} be a four dimensional real manifold modeling the spacetime manifold, with tangent bundle $T(\mathcal{M})$. Take in addition a vector bundle $A(\mathcal{M})$ whose fiber in each point is spin algebra. Take also a pointwise Pauli injection between $T(\mathcal{M})$ and $\text{Re}(\Lambda_{11}^*)(\mathcal{M})$. The gauged version of $\text{Aut}(A)$ shall be nothing but the product preserving vector bundle automorphisms of $A(\mathcal{M})$, and they act on $T(\mathcal{M})$ as the combined group

of diffeomorphisms and pointwise spacetime metric conformal rescalings, being the symmetries of (conformal) GR.

Appendix A.3. Meaning of dressing transformations

In the presented example the physical meaning of the nilpotent normal subgroup N can be understood as the “dressing” of pure one-particle states of a formal QFT model at a fixed spacetime point or momentum. Note, that spin algebra differs from a CAR algebra of QFT with the fact that the antiparticle creation operators are not yet identified with particle annihilation operators. It can be shown however [12], that an $\text{Aut}(A)$ -covariant family of self-dual CAR algebras can be associated to the spin algebra A , and vice-versa. Here, the self-dual CAR algebra is a mathematical structure, introduced by Araki [16], formally describing the algebraic behavior of quantum field operators. With the use of this relation, the spin algebra is a convenient reparametrization of the quantum field algebra of a QFT at a fixed point of spacetime or momentum space, revealing the hidden internal symmetry subgroup N . The details of the spin algebra \leftrightarrow self-dual CAR algebra family correspondence is, however, out of the scope of the present paper mainly focusing on unification, and shall be rather discussed in [12].

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